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ON THE ASYMPTOTIC THEORY OF SONIC FLOW OVER BODIES OF REVOLUTION *

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It is shown that the asymptotic theory of the distant field when applied to the problem of flow of perfect gas over a body of revolution, a flow that is sonic at infinity, can be supplemented by new terms containing arbitrary constants which carry information about the shape of the body.

The main feature of the flow of gas over a wing profile, when the flow velocity at infinity is sonic, is defined by the self-similar solution of transonic equations /1,2/. The axisymmetric analog of that solution was analyzed in /3/. Ryzhov and Shefter /4/ had noted that the difference between the plane and axisymmetric cases is that in the latter viscosity and thermal conductivity play an essential part. The allowance for their effect in the study of viscous gas flow over a finite body leads to the asymptotic theory of the distant field. A survey of achievements in this line of investigations appeared in /5/.

The development of an asymptotic theory applicable to inviscid gas is, nevertheless of some interest. This was done in /6,7/ for the plane flow, using expansion in the hodograph plane. Respective constructions for the plane /8/ and three-dimensional /9/cases were obtained using expansions in self-similar components directly in the flow plane. Some details of such expansions were discussed in /10,11/.

The asymptotic theory of the distant field uses the inverse expansion (in the terminology of /12/) whose characteristic feature is the appearance of indeterminate coefficients which are generally determined by the shape of the body. Their actual calculation is a difficult unsolved problem. Because of this the completeness of determination of the distant field in, for instance /8,9/, remained an open question: is the derived asymptotic solution valid for the flow over a body of arbitrary shape or only over bodies of a particular form?

Profiles for which the distant field defined by the expansion in /8/ is inapplicable in the plane case were indicated in /13/.

A similar construction derived below is valid in the case of axisymmetric flow. Since the question of inclusion of arbitrary constants in the asymptotic expansion is realted to some problem of proper solutions of the corresponding differential operator, the investigation is carried out on the basis of the transonic equation. Extension of the asymptotic theory of distant field /9,10/ is obtained at the cost of introduction the method of distorted coordinates /12/ and the inclusion in the expansion of logarithmic terms. The question of completeness of such extension remains open.

1. Let us consider the flow of perfect gas over a symmetric wing profile of infinite span or over a body of revolution at zero angle of attack. Let the velocity of the oncoming stream be sonic at infinity. We introduce a rectangular or cylindrical system of coordinates x and y, with the x-axis lying on the flow axis of symmetry. The motion of gas is, then, defined by the approximate system of transonic equations /2/

$$-uu_x + v_y + (\omega / y)\, v = 0,\ \ u_y - v_x = 0 \tag{1.1}$$

where u and v are dimensionless velocity components of a uniform sonic stream, and ω is a parameter which is zero or unity in a plane or axisymmetric flow, respectively.

The distant flow field between the negative semiaxis x and the limit characteristic is investigated. At the boundaries of that region the flow must satisfy conditions

$$u\,(x, y),\ v\,(x, y) \tag{1.2}$$

which are analytic functions at the limit characteristic

$$v\,(x, 0) = 0,\ x < 0 \tag{1.3}$$

System (1.1) with conditions (1.2) and (1.3) will be called Problem 1. Let us consider the self-similar solution of that problem

$$u = y^{2n-2}\, U_0\,(\zeta),\ v = y^{3n-3}\, V_0\,(\zeta),\ \zeta = xy^{-n} \tag{1.4}$$

where ζ is the self-similar variable, with $\zeta = -\infty$ corresponding to the negative semiaxis x, and $\zeta = \zeta_c$ to the limit characteristic (point ζ_c is determined by the condition $U_0\,(\zeta_c) = n^2\,(\zeta_c^2)$. For the plane flow exponent $n = {}^4/_5$ /1/ and for an axisymmetric one $n = {}^4/_7$ /3/.

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Velocities U_0 and V_0 were determined in /1,14/ in terms of algebraic functions.

The effect of the body in the stream is taken into account by representing the solution of Problem 1 in the form of expansions of the unknown functions u and v in self-similar components

$$u = y^{2n-2}[U_0(\zeta) + y^{h_1}U_1(\zeta) + y^{h_2}U_2(\zeta) + \ldots], \quad v = y^{3n-3}[V_0(\zeta) + y^{h_1}V_1(\zeta) + y^{h_2}V_2(\zeta) + \ldots] \tag{1.5}$$

where U_i and V_i $(i = 1, 2, \ldots)$ are the new unknown functions which satisfy ordinary linear differential equations and some boundary conditions at the ends $\zeta = -\infty$ and $\zeta = \zeta_c$. From (1.2) for the condition at point ζ_c we obtain

$$U_i = R(\zeta - \zeta_c), \; V_i = R(\zeta - \zeta_c)$$

Here and in what follows we denote functions that are analytic in the neighborhood $\varepsilon = 0$ by the symbol $R(\varepsilon)$.

If (1.5) is to determine the solution of Problem 1, exponents h_i must have definite values. As in /10/ we denote them by h_i^-

$$\omega = 0, \; h_i^- = -2i/5, \qquad \omega = 1, \; h_i^- = {}^1/_7\,[2i + 1 - (24i^2 + 24i + 1)^{1/2}]$$

In the asymptotic theory of distant field the boundary condition at the surface of the body is ignored, therefore a denumerable set of form parameters arises in expansions (1.5). These parameters are arbitrary constants which carry information on the form of the body but remain indeterminate for local study within $x^2 + y^2 \gg 1$. Varying the form parameters enables us to obtain a wide class of solutions of Problem 1. However not any solution of that problem can be expanded in series (1.5) with $h_i = h_i^-$. This conclusion is based on data in /13/, where a solution of Problem 1 somewhat different from (1.5) was obtained for the case of plane flow by the hodograph method. In the physical plane that solution is obtained by using the method of coordinate distortion /12/ in which not only the unknown functions u and v but, also, the independent variable ζ are expanded

$$u = y^{2n-2}[u_{00}(z) + y^{h_1}u_{10}(z) + y^{h_2}u_{20}(z) + \ldots], \quad v = y^{3n-3}[v_{00}(z) + y^{h_1}v_{10}(z) + y^{h_2}v_{20}(z) + \ldots] \tag{1.6}$$

$$\zeta = z + y^{h_1}\zeta_{10}(z) + y^{h_2}\zeta_{20}(z) + \ldots$$

where $z = -\infty$ corresponds to the negative x-axis and $z = z_c$ to the limit characteristic $(u_{00}(z_c) = n^2 z_c^2)$. The principal terms in (1.6) coincide with those of the self-similar solution (1.4)

$$u_{00}(z) = U_0(z), \; v_{00}(z) = V_0(z)$$

The spectrum of exponents $\{h_i\}$ obtained in /13/ comprises $\{h_i^-\}$ as a subset. This shows that the class of solutions (1.6) of Problem 1 is wider than (1.5), and that certain functions u_{i0} and v_{i0} nonanalytic at point z_c are admissible. Since singularities of the admissible functions u_{i0} and v_{i0} are of a special form, condition (1.2) is satisfied.

The aim of the present work is to obtain for axisymmetric flows a solution of Problem 1 similar to (1.6). Since the hodograph method is ineffective when $\omega = 1$, the solution is constructed directly in the x, y plane using the method of distorted coordinates of the form

$$u = y^{2n-2}\{u_{00} + y^{h_1}u_{10} + y^{h_2}(u_{21}\ln y + u_{20}) + y^{h_3}[u_{32}\cdot(\ln y)^2 + u_{31}\ln y + u_{30}] + \ldots\} \tag{1.7}$$

$$v = y^{3n-3}[v_{00} + y^{h_1}v_{10} + y^{h_2}(v_{21}\ln y + v_{20}) + \ldots], \qquad \zeta = z + y^{h_1}\zeta_{10} + y^{h_2}(\zeta_{21}\ln y + \zeta_{20}) + \ldots$$

where functions u_{ij}, v_{ij}, and ζ_{ij} depend on z. The exponents h_i are selected so that the singularities of functions u_{ij} and v_{ij} are of the same special form as that of u_{i0} and v_{i0} in (1.6), as $z \to z_c$. For simplicity we restrict (1.7) to series in which h_i form a decreasing arithmetical progression

$$h_i = ih_1, \; h_1 < 0, \; i = 1, 2, \ldots$$

2. Let us, first, consider the plane flow $\omega = 0$ and $n = {}^4/_5$. To obtain some preliminary information for constructing solution (1.7) with $\omega = 1$ we present certain results from /13/ and their corollaries.

In the hodograph plane system (1.1) becomes linear

$$-uy_v + x_u = 0, \; y_u - x_v = 0 \tag{2.1}$$

Let us consider the solution of system (2.1) as the sum of two self-similar terms

$$y = v^{-4/3}Y_0(t) + v^{1/6}Y_1(t), \; x = v^{-4/3}X_0(t) + v^{1/2}X_1(t), \qquad t = v/(v^2 - {}^4/_9\,u^3)^{1/2} \tag{2.2}$$

Terms with subscript 0 represent the solution /1/ of the problem of flow over a profile of infinite span, those with subscript 1 the solution /15/ that defines the second asymptotic type of flow in plane Laval nozzles. The value $t = 0$ corresponds to the negative semiaxis x and $t = \infty$ to the limit characteristic. By inverting functions $x, y \mid u, v$ in (2.2) we obtain $u, v \mid x, y$ as the solution of Problem 1. This solution is of the form (1.6) with exponents $h_i = -11i/10$. Functions u_{i0}, v_{i0}, and ζ_{i0} can be expressed in terms of parameter t

$$u_{i0} = u_{i0}^w(t),\ v_{i0} = v_{i0}^w(t),\ \zeta_{i0} = \zeta_{i0}^w(t),\ z = z^w(t) \tag{2.3}$$

where $u_{i0}^w, \ldots, z^w$ are an algebraic combinations of the known functions Y_0, Y_1, X_0, and X_1 /13/.

Let us determine the equations and boundary conditions which must be satisfied by u_{i0} and v_{i0} in the interval $-\infty < z < z_c$ if (1.6) is to represent the solution of Problem 1, and, also, the behavior of the distorting functions ζ_{i0} at the ends of that interval. Substituting (1.6) into (1.1) we obtain a sequence of systems of linear differential equations

$$\begin{gathered} -u_{00}u_{i0}' - u_{00}'u_{i0} + (3n - 3 + h_i)v_{i0} - nzv_{i0}' = K_{i0} \\ (2n - 2 + h_i)u_{i0} - nzu_{i0}' - v_{i0}' = L_{i0},\ i = 1, 2, \ldots \end{gathered} \tag{2.4}$$

where the prime denotes differentiation with respect to the argument, and the right-hand sides K_{i0} and L_{i0} depend on u_{m0}, v_{m0}, and ζ_{m0} and their first derivatives. Boundary conditions for (2.4) are obtained with the use of the exact solution (2.3). Making $t \to \infty$ we find that as $z \to z_c$

$$u_{i0} = R(\Delta),\quad v_{i0} = R(\Delta) \tag{2.5}$$

$$\zeta_{i0} = R(\Delta),\ \Delta = (z - z_c)^{1/3} \tag{2.6}$$

At the limit as $t \to 0$ we obtain that in the neighborhood $z = -\infty$

$$u_{i0} = Z^{-2n+2-h_i}R(Z^2),\ v_{i0} = Z^{-3n+4-h_i}R(Z^2) \tag{2.7}$$

$$\zeta_{i0} = Z^{-n-h_i}R(Z^2),\ Z = (-z)^{-1/n} \tag{2.8}$$

Problem 1 for the unknown functions u and v is, thus, transformed into Problem 2 with u_{i0} and v_{i0} as the new unknown functions, and consisting of system (2.4) with conditions (2.5) and (2.7). The purpose of the distorting functions ζ_{i0} is to prevent an increase of singularities in u_{i0} and v_{i0} in the neighborhood of point z_c with the increase of the approximation number /12/. Functions ζ_{i0} must be selected with allowance for the requirements of (2.6) and (2.8).

3. Let us now consider axisymmetric flows with $\omega = 1$ and $n = 4/7$. We shall seek a solution of Problem 1 of the form (1.7), where for the representative velocities we shall formulate some Problem 3 similar to Problem 2. As shown by equalities (2.6) and (2.8) the distorting functions are of a fairly complex form. To simplify them and ensure the validity of respective expansions as $y \to 0$, we pass from variables z and y to the new independent variables τ and ρ which we introduce using the data in /9/. In these variables the boundary condition at the axis of symmetry is of simpler form.

Representative u_{00} and v_{00} are of the form

$$u_{00} = 2^4 \cdot 7^{-2}(6\tau - 5)\tau^{3/7},\quad v_{00} = 3 \cdot 2^6 \cdot 7^{-3}(-4\tau + 5)\tau^{8/7},\qquad z = 7^{-1}(12\tau - 5)\tau^{-2/7}$$

The value $\tau = 0$ corresponds to the negative semiaxis x and $\tau = 1$ to the limit characteristic. We define the variable ρ by $\rho = y\tau^{-1/2}$. In new variables expansion (1.7) assumes the form

$$\begin{gathered} u = \rho^{-6/7}\{f_{00}(\tau) + \rho^{h_1}f_{10}(\tau) + \rho^{h_2}[f_{21}(\tau)\ln\rho + f_{20}(\tau)] + \ldots\} \\ v = \rho^{-9/7}\{g_{00}(\tau) + \rho^{h_1}g_{10}(\tau) + \rho^{h_2}[g_{21}(\tau)\ln\rho + g_{20}(\tau)] + \ldots\} \\ \zeta = \tau^{-2/7}\{\xi_{00}(\tau) + \rho^{h_1}\xi_{10}(\tau) + \rho^{h_2}[\xi_{21}(\tau)\ln\rho + \xi_{20}(\tau)] + \ldots\} \end{gathered} \tag{3.1}$$

where the principal terms are determined by the equalities

$$f_{00} = 2^4 \cdot 7^{-2}(6\tau - 5),\ g_{00} = 3 \cdot 2^6 \cdot 7^{-3}\tau^{1/2}(-4\tau + 5),\qquad \xi_{00} = 7^{-1}(12\tau - 5)$$

Substituting (3.1) into (1.1) we obtain for the representative f_{ij} and g_{ij} a sequence of systems of linear differential equations

$$\begin{gathered} M_i(f_{ij}, g_{ij}) \equiv m_1 f_{ij}' + m_2 f_{ij} + m_3 g_{ij}' + m_4 g_{ij} = \mu_{ij}, \\ N_i(f_{ij}, g_{ij}) \equiv n_1 f_{ij}' + n_2 f_{ij} + n_3 g_{ij}' + n_4 g_{ij} = \nu_{ij} \\ m_1 = -8T^{-1}\tau(6\tau - 5),\ m_2 = 4/7\,T^{-1}[7h_i(6\tau - 5) - 156\tau + 60] \\ m_3 = -2T^{-1}(12\tau - 5)\tau^{1/2},\ m_4 = T^{-1}\tau^{-1/2}(42h_i\tau - 24\tau + 5) \\ n_1 = -2T^{-1}(12\tau - 5),\ n_2 = 42T^{-1}(-6/7 + h_i) \\ n_3 = -49/2T^{-1}\tau^{1/2},\ n_4 = 49/4T^{-1}\tau^{-1/2}(-9/7 + h_i),\quad T = 30\tau + 5 \end{gathered} \tag{3.2}$$

The right-hand sides μ_{ij} and ν_{ij} depend on preceding approximations

$$\mu_{10} = \tau^{-1/2}[2/7\,g_{00}D\xi_{10} + (4/7 + h_1)\xi_{10}Dg_{00}],\qquad \nu_{10} = \tau^{-1}[6/7\,f_{00}D\xi_{10} + (4/7 + h_1)\xi_{10}Df_{00}]$$

$$\mu_{21} = \tau^{-1/2}[2/7\,g_{00}D\xi_{21} + (4/7 + h_2)\xi_{21}Dg_{00}],\qquad \nu_{21} = \tau^{-1}[6/7\,f_{00}D\xi_{21} + (4/7 + h_2)\xi_{21}Df_{00}]$$

$$\mu_{20} = f_{10}Df_{10} + \tau^{-1/2}\{{}^2/_7 g_{00}D\xi_{20} + ({}^2/_7 - h_1)g_{10}D\xi_{10} - g_{21} + ({}^4/_7 + h_1)\xi_{10}Dg_{10} + [({}^4/_7 + h_2)\,\xi_{20} + \xi_{21}]\,Dg_{00}\} - 49/4T^{-1}\,[f_{00}f_{21} + {}^2/_7\tau^{-1/2}\,g_{00}\xi_{21} + {}^4/_7\xi_{00}\tau^{-1/2}g_{21}]$$

$$\nu_{20} = \tau^{-1}\{{}^6/_7 f_{00}D\xi_{20} + ({}^6/_7 - h_1)\,f_{10}D\xi_{10} - f_{21} + ({}^4/_7 + h_1)\,\xi_{10}Df_{10} + [({}^4/_7 + h_2)\xi_{20} + \xi_{21}]\,Df_{00} - 49/4T^{-1}\,[{}^6/_7 f_{00}\xi_{21} + {}^4/_7\xi_{00}f_{21} + \tau^{1/2}\,g_{21}]\},\ldots$$

where the operator D is determined as

$$Df_{ij} = 49\,/\,2T^{-1}\,[-{}^1/_2\,(-\,{}^6/_7 + h_i)f_{ij} + \tau f_{ij}']$$

$$Dg_{ij} = 49\,/\,2T^{-1}\,[-{}^1/_2\,(-\,{}^9/_7 + h_i)\,g_{ij} + \tau g_{ij}']$$

$$D\xi_{ij} = 49\,/\,2T^{-1}\,[-{}^1/_2\,({}^4/_7 + h_i)\,\xi_{ij} + \tau\xi_{ij}']$$

Besides (3.2) we consider the homogeneous system

$$M_i\,(F_i\,(\tau), G_i\,(\tau)) = 0, \quad N_i\,(F_i\,(\tau), G_i\,(\tau)) = 0 \tag{3.3}$$

We assume the distorting functions ξ_{ij} to be of the form

$$\xi_{ij} = a_{ij} + b_{ij}\,(\tau - 1)^{1/3}$$

where the constants a_{ij} and b_{ij} are selected so as to avoid an increase of singularities of representative velocities.

We define functions f_{ij} and g_{ij} as follows:

$$f_{ij} = C_{ij}F_i + f_{ij}^p, \quad g_{ij} = C_{ij}G_i + g_{ij}^p \tag{3.4}$$

where C_{ij} is an arbitrary constant and the index p denotes the particular solution of the inhomogeneous system (3.2).

Boundary conditions for the unknown functions at the ends $\tau = 0$ and $\tau = 1$ are independent on the condition of symmetry (1.3) which is transformed into the stipulation that as $\tau \to 0$

$$F_i = R\,(\tau), \quad G_i = \tau^{1/2}R\,(\tau) \tag{3.5}$$

$$f_{ij}^p = R\,(\tau), \quad g_{ij}^p = \tau^{1/2}\,R\,(\tau) \tag{3.6}$$

We rewrite the condition (1.2) at the limit characteristic by analogy with (2.5) in the form

$$f_{ij} = R\,(\delta), \quad g_{ij} = R\,(\delta), \quad \delta = (\tau - 1)^{1/3}, \ \tau \to 1 \tag{3.7}$$

System (3.2) and conditions (3.5)—(3.7) constitute Problem 3 for the representative f_{ij} and g_{ij}.

4. Let us first, consider functions F_i and G_i. We transform system (3.3) into a hypergeometric equation by introducing the substitution

$$F_i = \eta_1 Q_i\,(\tau) + \eta_2 Q_i'\,(\tau), \quad G_i = \eta_3\,Q_i\,(\tau) + \eta_4 Q_i'\,(\tau) \tag{4.1}$$

$$\eta_1 = -\,49\,/\,4T^{-1}\,(-{}^2/_7 + h_i), \quad \eta_2 = 49\,/\,2T^{-1}\tau$$

$$\eta_3 = 42T^{-1}\,(-{}^2/_7 + h_i)\,\tau^{1/2}, \quad \eta_4 = -2T^{-1}\,(12\tau - 5)\tau^{1/2}$$

and for function Q_i obtain the equation /9/

$$\tau\,(1 - \tau)Q_i'' + [\gamma_i - \tau\,(\alpha_i + \beta_i + 1)]\,Q_i' - \alpha_i\beta_i Q_i = 0 \tag{4.2}$$

$$\alpha_i = 7/10\,({}^5/_7 + h_i + W_i), \quad \beta_i = 7/10\,({}^5/_7 + h_i - W_i)$$

$$\gamma_i = 1, \quad W_i = [1 + 24/7\,(-\,{}^2/_7 + h_i) + 6\,(-{}^2/_7 + h_i)^2]^{1/2}$$

In conformity with condition (3.5) we assume the solution of this equation to be of the form

$$Q_i = F\,(\alpha_i, \beta_i, \gamma_i; \tau) \tag{4.3}$$

where F is the symbol of the hypergeometric function. The second fundamental solution of Eq. (4.2) is unsuitable, since its expansion in the neighborhood of $\tau = 0$ contains logarithmic terms.

Let us now consider functions f_{ij}^p and g_{ij}^p. If $j = i - 1$, the particular solution satisfying (3.6) is of the simple form

$$f_{i,i-1}^p = \xi_{i,i-1}Df_{00}, \quad g_{i,i-1}^p = \xi_{i,i-1}\,Dg_{00} \tag{4.4}$$

When $j \neq i - 1$ the presentation of partial solution is difficult. The analysis of the right-hand sides of μ_{ij} and ν_{ij} of system (3.2) shows that when preceding approximations appearing in them satisfy (3.5) and (3.6), functions μ_{ij} and ν_{ij} are analytic in the neighborhood of $\tau = 0$

$$\mu_{ij} = R(\tau),\ \nu_{ij} = R(\tau)$$

Hence the particular solutions f_{ij}^p and g_{ij}^p which satisfy (3.6) exist. They can be obtained, for instance, in the form of expansions in the neighborhood of $\tau = 0$, with the expansion coefficients determined by the substitution into (3.2). The boundary conditions (3.5) and (3.6) can, thus, be satisfied. Note that no constraints whatsoever are imposed on the quantities

$$h_i,\ a_{ij},\ b_{ij},\ C_{ij} \tag{4.5}$$

5. Let us now pass to the boundary condition (3.7) for whose fulfillment we shall choose constants (4.5) in a specific manner.

To determine indices h_i^- we consider the analytic continuation of (4.3) to the neighborhood of point $\tau = 1$

$$Q_i = A_i F(\alpha_i, \beta_i, -s_i; 1-\tau) + B_i Q_i^*, \qquad Q_i^* = (1-\tau)^{s_i+1} F(\gamma_i - \alpha_i, \gamma_i - \beta_i, s_i + 2; 1-\tau) \tag{5.1}$$

$$A_i = \Gamma(\gamma_i)\Gamma(s_i+1)/[\Gamma(\gamma_i - \alpha_i)\Gamma(\gamma_i - \beta_i)], \qquad B_i = \Gamma(\gamma_i)\Gamma(-s_i - 1)/[\Gamma(\alpha_i)\Gamma(\beta_i)]$$

where Γ denotes the gamma function and s_i is defined by

$$s_i = \gamma_i - \alpha_i - \beta_i - 1 = -{}^7/_5 h_i - 1 \tag{5.2}$$

Using (4.1) and (5.1) we find that as $\tau \to 1$

$$F_i, G_i = R(\tau - 1) + (\tau - 1)^{s_i} R(\tau - 1) \tag{5.3}$$

Let us consider functions F_1 and G_1 setting $s_1 = {}^1/_3$. From (5.2) we have $h_1 = -{}^{20}/_{21}$. Note that $R(\tau - 1) = R(\delta^3)$, $\delta = (\tau - 1)^{1/3}$. Then from (5.3) follows that

$$F_1, G_1 = R(\delta^3) + \delta R(\delta^3)$$

from which $F_1, G_1 = R(\delta)$. We take the particular solution f_{10}^p, g_{10}^p is taken in the form (4.4). Obviously $f_{10}^p, g_{10}^p = R(\delta)$. Consequently functions f_{10} and g_{10} which satisfy Problem 3 are determined by (3.4) in which constant C_{10} is arbitrary.

Restricting the analysis to series (3.1) in which indices h_i represent a decreasing arithmetical progression, we set

$$h_i = -20i/21,\quad i = 1, 2, \ldots \tag{5.4}$$

Let us consider higher approximations of f_{10} and g_{10}, and begin by analyzing functions Q_i. From (5.2) and (5.4) we have

$$s_i = 4i/3 - 1,\quad i = 1, 2, \ldots \tag{5.5}$$

The analysis of quantities s_i in (5.5) and β_i in (4.2) shows that the analytic continuation of function (4.3) into the neighborhood of $\tau = 1$ may lead to the following three cases.

a) Conventional case in which exponent s_i is not a positive integer and parameter β_i not a negative integer $(i \neq 3, 6, 9, \ldots)$.

b) Degenerate case in which s_i is a positive integer, and β_i is a negative integer, for instance, when $i = 15$ $s_i = 19$ and $\beta_i = -34$. In this case the hypergeometric function is a polynomial whose analytic continuation yields /16/

$$Q_i = (-1)^m (s_i + 2)_m (m!)^{-1} Q_i^*, \qquad m = \alpha_i - \gamma_i,\quad (a)_m = \Gamma(a + m)/\Gamma(a) \tag{5.6}$$

Taking into account the form (5.5) of exponents s_i and equalities (4.1), (5.1), and (5.6) we conclude that in cases a) and b)

$$F_i, G_i = R(\delta)$$

c) Logarithmic case in which s_i is a positive integer, and parameter β_i is not a negative integer $(i = 3, 6, 9, 12, 18, \ldots)$. In this case the analytic continuation of function (4.3) is of the form /16/

$$Q_i = A_i \sum_{m=0}^{s_i} \frac{(\alpha_i)_m (\beta_i)_m}{(-s_i)_m m!}(1-\tau)^m + \frac{\Gamma(\gamma_i)(-1)^{s_i+1}}{\Gamma(\alpha_i)\Gamma(\beta_i)} \times \sum_{m=0}^{\infty} \frac{(\gamma_i - \alpha_i)_m (\gamma_i - \beta_i)_m}{(m+1+s_i)!\, m!} (\varphi_m - \ln|1-\tau|)(1-\tau)^{m+s_i+1}$$

$$\varphi_m = \psi(m+1) + \psi(m + s_i + 2) - \psi(\alpha_i + m + 1 + s_i) - \psi(\beta_i + m + 1 + s_i),\quad \psi(a) = \Gamma'(a)/\Gamma(a)$$

and, consequently, as $\tau \to 1$

$$F_i = R(\delta^3) + \Omega_i F_i^* \ln|\tau - 1|,\ G_i = R(\delta^3) + \Omega_i G_i^* \ln|\tau - 1| \tag{5.7}$$

$$F_i^* = \eta_1 Q_i^* + \eta_2 Q_i^{*\prime},\quad G_i^* = \eta_3 Q_i^* + \eta_4 Q_i^{*\prime},\qquad \Omega_i = \Gamma(\gamma_i)(-1)^{s_i}/[\Gamma(\alpha_i)\Gamma(\beta_i)(s_i+1)!]$$

6. Let us consider the representative f_{2j} and g_{2j} ($s_2 = {}^5/_3$ in the conventional case). Taking into consideration the data in Sects. 4 and 5 we conclude that functions f_{21} and g_{21} which satisfy Problem 3 are of the form (4.3) in which F_2, G_2 and f_{21}^p, g_{21}^p are determined

by (4.1) and (4.4), respectively.

Let us consider functions f_{20} and g_{20}. Since the derivation of the particular solution f_{20}^p and g_{20}^p is difficult, we shall investigate the behavior of functions in the neighborhood of $\tau = 1$ using the expansion in series with $\tau \to 1$. We transform system (3.2) by eliminating g_{ij}' from the first equation

$$(\tau - 1)\, l_1 f_{ij}' + l_2 f_{ij} + l_4 g_{ij} = \lambda_{ij}, \quad N_i(f_{ij}, g_{ij}) = \nu_{ij} \tag{6.1}$$

$$l_1 = -40/49, \quad l_2 = -{}^4/_7\,[-{}^6/_7 + h_i + (120\tau - 30)\, T^{-1}]$$

$$l_4 = (-{}^2/_7 + h_i)\, \tau^{-1/2}, \quad \lambda_{ij} = -4/49\,(12\tau - 5)\nu_{ij} + \mu_{ij}$$

The right-hand sides λ_{20} and ν_{20} of syatem (6.1) depend on $f_{00}, f_{10}, \ldots$ and their first derivatives. Consequently when $\tau \to 1$ we have the expansion

$$\lambda_{20} = \sum_{m=-2}^{\infty} (\tau - 1)^{m/3} \lambda_{20}^{(m/3)}, \quad \nu_{20} = \sum_{m=-2}^{\infty} (\tau - 1)^{m/3} \nu_{20}^{(m/3)}$$

The analysis of system (6.1) show that, if the expansion of functions f_{20} and g_{20} in the neighborhood of $\tau = 1$ is to be free of terms of order $(\tau - 1)^{-2/3}$ and $(\tau - 1)^{-1/3}$, it is necessary to stipulate

$$\lambda_{20}^{-2/3} = \lambda_{20}^{-1/3} = 0 \tag{6.2}$$

After some simple transformations in the right-hand side λ_{20}, it is possible to separate the group of terms that generate the singularity increase

$$\lambda_{20} = (Df_{10} - D\xi_{10} Df_{00})\, [(16/49\, \tau^{-1} \xi^2_{00} - f_{00}) D\xi_{10} - ({}^4/_7 + h_1)\, \tau^{-1}\, ({}^8/_7)\, \xi_{00}\xi_{10} + f_{10}] + \ldots \tag{6.3}$$

The analysis of (6.3) shows that condition (6.2) is satisfied when the coefficients of the distorting function ξ_{10} are taken in the form

$$a_{10} = -3 \cdot 7^2 \cdot 320^{-1}\, (-{}^2/_7 + h_1)^2\, ({}^5/_7 + h_1)^{-1} A_1 C_{10}, \quad b_{10} = -\,7^3 \cdot 600^{-1} B_1 C_{10}$$

7. We shall now prove that for any i and j there exist functions f_{ij} and g_{ij} which satisfy Problem 3. Let us assume that preceding approximations have been already obtained from the solution of Problem 3 and that the condition of nonincrease of singularities

$$\lambda_{ij} = \sum_{m=0}^{\infty} (\tau - 1)^{m/3} \lambda_{ij}^{(m/3)}, \quad \nu_{ij} = \sum_{m=-2}^{\infty} (\tau - 1)^{m/3} \nu_{ij}^{(m/3)} \tag{7.1}$$

is satisfied.

Let us consider the behavior of the general integral of system (6.1) in the neighborhood of $\tau = 1$. Note that $\tau = 1$ defines a regular singular point for system (6.1) and the numbers 0 and s_i are roots of the respective characteristic equation. Since s_i is a fraction in the form of a positive integer divided by three, and the right-hand sides of system (6.1) expand in series in powers of $(\tau - 1)^{m/3}$, and m is an integer, hence as $\tau \to 1$ we have the expansion

$$f_{ij} = \sum_{m=0}^{\infty} (\tau - 1)^{m/3} f_{ij}^{(m/3)} + \omega_{ij} F_i^* \ln|\tau - 1|, \quad g_{ij} = \sum_{m=0}^{\infty} (\tau - 1)^{m/3} g_{ij}^{(m/3)} + \omega_{ij} G_i^* \ln|\tau - 1| \tag{7.2}$$

where ω_{ij} is some coefficient, $f_{ij}^{(0)}$, $f_{ij}^{(s_i)}$ are arbitrary constants, and the remaining coefficients are determined by recurrent formulas in terms of these. For some particular values of constants $f_{ij}^{(0)}$ and $f_{ij}^{(s_i)}$ expansions (7.2) define the behavior of solution (3.4)—(3.6) in which we are interested. Formulas (7.2) show that for satisfying the boundary condition (3.7) it remains to specify

$$\omega_{ij} = 0 \tag{7.3}$$

If $j = i - 1$ and $i \neq 3, 6, 9, 12, 18, \ldots$, condition (7.3) is automatically satisfied. In the remaining cases (7.3) is to be considered as the equation concerning constants C_{im} in representative velocities.

To determine coefficient ω_{ij} we expand functions $l_k(\tau)$ and $n_k(\tau)$, $k = 1, 2, 3, 4$ with $\tau \to 1$

$$l_k = \sum_{m=0}^{\infty} l_k^{(m)} (\tau - 1)^m, \quad n_k = \sum_{m=0}^{\infty} n_k^{(m)} (\tau - 1)^m \tag{7.4}$$

The substitution of (7.1), (7.2), and (7.4) into (6.1) yields

$$\omega_{ij} = {}^5/_2 (-1)^{s_i+1} (s_i^2 + s_i)^{-1} \{n_3^{(0)} s_i\, [\lambda_{ij}^{(s_i)} - \Sigma_1] - l_4^{(0)}\, [\nu_{ij}^{(s_i-1)} - \Sigma_2]\}$$

$$\Sigma_1 = \sum_{m=1}^{\sigma} [l_2^{(m)} f_{ij}^{(s_i-m)} + l_4^{(m)} g_{ij}^{(s_i-m)}], \quad \Sigma_2 = \sum_{m=1}^{\sigma} (s_i - m)\, [n_1^{(m)} f_{ij}^{(s_i-m)} + n_3^{(m)} g_{ij}^{(s_i-m)}] + \sum_{m=0}^{\sigma-1} [n_2^{(m)} f_{ij}^{(s_i-1-m)} + n_4^{(m)} g_{ij}^{(s_i-1-m)}]$$

where δ denotes the integral part of number s_i.

8. Let us show on specific examples in what manner can Eq. (7.3) be satisfied.

Let us consider the representative f_{20} and g_{20}, and write condition (7.3) as follows:

$$n_3^{(0)} s_2 [\lambda_{20}^{(s_2)} - \Sigma_1] - l_4^{(0)} [\nu_{20}^{(s_2-1)} - \Sigma_2] = 0 \tag{8.1}$$

The right-hand side λ_{20} of system (6.1) includes functions f_{21} and g_{21} which contain the arbitrary constant C_{21}. Hence the coefficient $\lambda_{20}^{(s_2)}$ can be represented as

$$\lambda_{20}^{(s_2)} = {}^4/_5 (s_2 + 1)(-1)^{s_2+1} B_2 C_{21} + \ldots$$

where the dots indicate terms free of C_{21}. The analysis of expressions Σ_1 and Σ_2 and of coefficient $\nu_{20}^{(s_2-1)}$ shows that constant C_{21} does not appear in them, which means that (8.1) is a linear equation with respect to C_{21}.

Let us consider representative f_{3j} and g_{3j} ($s_3 = 3$ represents the logarithmic case). Taking into account (5.7) we represent coefficient ω_{3j} in the form

$$\omega_{3j} = C_{3j}\Omega_3 + \ldots$$

where the dots indicate terms free of C_{3j}. Equation (3.7) obviously represents a linear equation for C_{3j}. Continuation of the described process of constructing solutions f_{ij} and g_{ij} shows that a part of constants C_{ij} must assume in conformity with condition (7.3) fixed values. Only constants C_{i0}, $i \neq 3, 6, 9, 12, 18, \ldots$ remain arbitrary.

Let us show that the constructed solution (3.1) satisfies condition (1.2), noting that functions

$$u = u(\delta, \rho), \quad v = v(\delta, \rho), \qquad x = \rho^{4/7}(\xi_{00} + \rho^{h_1}\xi_{10} + \ldots), \quad y = \rho(1 + \delta^3)^{1/2} \tag{8.2}$$

are analytic by construction for small σ and finite ρ. Let us examine the Jacobian of transformation $J(\delta, \rho) = \partial(x, y) / \partial(\delta, \rho)$ at the limit characteristic $\delta = 0$

$$J(0, \rho) = \partial x(0, \rho) / \partial\delta = \rho^{4/7} [b_{10}\rho^{h_1} + O(\rho^{h_2} \ln \rho)]$$

Since $J \neq 0$, the inverse functions of $\delta, \rho \mid x, y$ are analytic in the limit characteristic neighborhood. Substituting these into (8.2) we obtain the velocity field with property (1.2).

Note that expansions (3.1) may be supplmented by terms with index h_i^-. The obtained solution is more general, since it contains (3.1) and (1.5) as particular cases.

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